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Exact results for vicious walker models of domain walls

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Abstract. Non-intersecting (or vicious) random walker models in one dimension can be interpreted as models of domain walls in two dimensions. Three problems pertaining to vicious walker models are solved. The first is the exact evaluation of the partition function for the random turns model of vicious walkers on a lattice. In this model, at each tick of the clock, a randomly chosen walker must move one step to the left or one step to the right. The second problem is the calculation of the mean spacing between walls in terms of the chemical potential for a Brownian motion model of continuous domain walls in a strip, while the final problem solved is the calculation of the correlation between defects, which occur when two domain walls meet and end without crossing the whole system.

1. Introduction

1.1. Domain walls

Phenomological theories of domain walls have been successful in predicting transitions between striped phases and pure phases in two-dimensional anisotropic lattice systems (see, e.g., [1] for a review). Each domain wall is modelled as a single piecewise straight line fixed at both ends of the system in one direction, with the wall progressing from one end of the system to the other without going backwards. Each wall has an energy associated with its precise shape, while the only interaction between walls is the condition that the walls cannot intersect.

In the majority of the literature (e.g. [1] and references therein) such models of domain walls are studied via a mapping to the many body quantum mechanical problem of the time evolution of free fermions on a lattice. However, use of this quantum mechanical analogue is not necessary as has been demonstrated by Fisher [2] in his treatment of domain walls in terms of vicious (meaning non-intersecting) random walkers. This viewpoint has allowed the discovery of many new results [2]-[9].

Two discrete models were formulated by Fisher [2]: the lock step model and the random turns model. In the continuum limit, both these models reduce to a model of impenetrable particles undergoing Brownian motion on a line. The precise detail of the discrete models was chosen so as to allow an exact evaluation of the corresponding partition functions, although this latter task was not carried out in full in [2].

1.2. Definitions of the lock step and random turns models

In both models there are N walkers confined to a one-dimensional lattice which may be either infinite, semi-infinite, finite or circular. For the lock step model, at each tick of the clock each walker must move one step to the left or one step to the right, with

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weightings w_{-1} and w_1 respectively, and obey the constraint that two walkers cannot occupy the same site (figure 1). For the random turns model, at each tick of the clock only a single randomly chosen walker moves one step to the left or one step to the right, with weightings w_{-1} and w_1 respectively. Again there is the constraint that two walkers cannot occupy the same site (see figure 2).

The partition function $Z_n^{(X)}(l_1, \ldots, l_N | l'_1, \ldots, l'_N)$ (X denotes the particular model: random turns (rt) or lock step (ls)) is given by the weighted sum of all allowed paths from $l_1, \ldots, l_N$ to $l'_1, \ldots, l'_N$ in n time steps. The weighting of each configuration is the product of the weightings of the individual paths. For the lock step model on a circular lattice, the exact evaluation of the partition function can be found in [7], while for the same model on a semi-infinite and finite lattice the partition function is evaluated in [6].

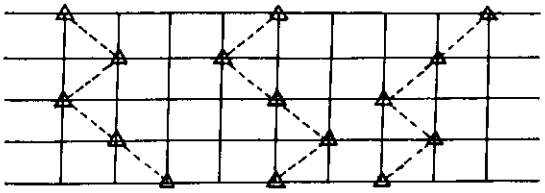

Figure 1. A typical configuration in the lock step model.

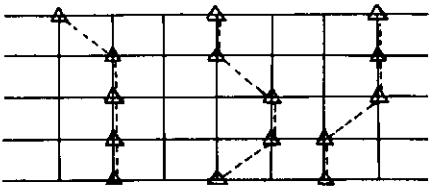

Figure 2. A typical configuration in the random turns model.

1.3. Aim and summary

In this paper three distinct problems pertaining to vicious random walkers are addressed. In section 2 the partition function for the lock step model on an infinite lattice is evaluated exactly. From the exact expression it is shown explicitly that for a fixed number of walkers the large step behaviour of the partition function corresponds to the partition function for impenetrable Brownian particles on a line. This proves that the large-time behaviour of probabilistic quantities (e.g. probability of survival, probability of a reunion etc) are the same for both models as claimed in [2] and [3]. Furthermore, with $w_{-1} = 0$ and $w_1 = 1$ the exact evaluation of the partition function gives the enumerations for a counting problem which occurs in a graphical expansion of the Hubbard model in the atomic limit [10].

In section 3 continuous domain walls, which are modelled by impenetrable Brownian particles, of finite length across an infinite strip are considered. By minimizing the free energy with respect to the average spacing between walls ν, it is found that there is a critical chemical potential σ_c (dependent on the width t of the strip) below which ν is infinite. Furthermore, with

$$\sigma = \varepsilon + \sigma_c \tag{1.1}$$

it is shown that as $\varepsilon \to 0^+$, ν diverges according to the formula

$$\varepsilon t \sim (2\nu^2/Dt)\, \mathrm{e}^{-\nu^2/Dt} \tag{1.2}$$

where D is the diffusion constant of the Brownian particles. Thus there is a phase transition from a striped phase to a pure phase with an essential singularity in the order parameter at the transition point.

In section 4, dislocation configurations within the Brownian motion model of domain walls are considered. Dislocations occur when two domain walls meet (or come into close proximity) and end without crossing the whole system. They model the configurations in the original lattice system responsible for Kosterlitz-Thouless type transitions (see e.g. [11] and references therein). The dislocations can be divided into two types according to their originating from the top or the bottom of the system. At large separation, the correlation $\rho^{(u)}(x^*, t^*)$ where $x^* := |x_1 - x_2|$ and $t^* := |t_1 - t_2|$, between two unlike dislocations at (x_1, t_1) and (x_2, t_2) exhibits the behaviour

$$\rho^{(u)}(x^*, t^*) \sim \frac{1}{\nu^2} \frac{1}{(x^*)^2 + (\pi D t^*/\nu)^2} \tag{1.3}$$

while the correlation $\rho^{(l)}(x^*, t^*)$ between two like dislocations behaves as

$$\rho^{(l)}(x^*, t^*) \sim \left(\frac{2\pi}{L}\right)^4 \frac{1}{\nu^2} [(x^*)^2 + (\pi D t^*/\nu)^2] \tag{1.4}$$

where L is the length of the system. The result (1.3) has been derived using fermion methods [11], while the behaviour (1.4) of the correlation between like defects, although often quoted (see e.g. [11]) does not seem to have been derived before.

2. The random turns model

2.1. A related lock step model

It was shown in [7] that the partition function for the lock step model defined above could be specified as the unique solution of a multi-dimensional difference equation. This is again true of the lock step model. Although the difference equation approach provides a proof of the formula for the partition function, it is a verification of the formula rather than a derivation. In this subsection we will provide a derivation of the formula, which can then be verified using the difference equation.

We first consider a (generalized) lock step model in which each walker can take a step to the left (weight w_{-1}), a step to the right (weight w_1) or stay stationary (weight w_0). With the walkers free to move on an infinite line, the partition function for a single walker going from site l_j' to site l_k in n steps is

$$Q_n^{(0)}(l_j'|l_k) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}(l_k - l_j')\theta} \phi(\theta)\, \mathrm{d}\theta \tag{2.1}$$

where

$$\phi(\theta) = (w_{-1}\, \mathrm{e}^{-i\theta} + w_0 + w_1\, \mathrm{e}^{i\theta})^n \tag{2.2}$$

(see e.g. [2]). Note that it is possible to locate the steps at which the walker does not move if we define $Q_n^{(0)}(l_j'|l_k)$ as in (2.1) but with

$$\phi(\theta) = \prod_{l=1}^{n} (w_{-1}\, \mathrm{e}^{-i\theta} + w_0^{(l)} + w_1\, \mathrm{e}^{i\theta}). \tag{2.3}$$

The occurrence of a weight $w_0^{(j)}$ in a term in the partition function tells us for that particular configuration the walker did not move at step j.

Let us now consider two walkers moving from l_1' and l_2' to l_1 and l_2 according to the rules of the generalized lock step model and with the condition that two walkers cannot arrive at the same site or pass one another. The partition function for this model does not seem to have a simple structure. However, the expression

$$\det\begin{bmatrix} Q_n^{(0)}(l_1'|l_1) & Q_n^{(0)}(l_1'|l_2) \\ Q_n^{(0)}(l_2'|l_1) & Q_n^{(0)}(l_2'|l_2) \end{bmatrix} \tag{2.4}$$

which is the exact partition function for $w_0 = 0$ [7], satisfies the same difference equation as the exact partition function and also vanishes if two walkers should arrive at the same site (it does not vanish if two walkers should pass each other without first intersecting).

If we assume that (2.4) contains all the configurations of the random turns model (this is certainly true of the exact partition function), then choosing $\phi(\theta)$ as (2.3) gives that the weighting of the configurations of the random turns model are the coefficient of

$$\prod_{l=1}^{n} w_0^{(l)} \tag{2.5}$$

in the multinomial expansion of the partition function in terms of $w_0^{(1)}, \ldots, w_0^{(n)}$. Thus we conjecture that

$$Z_n^{(\mathrm{rt})}(l_1', l_2'|l_1, l_2) = \left(\prod_{k=1}^{n} \int_0^1 \mathrm{d}x_k \, \mathrm{e}^{-ix_k} \right) \det\begin{bmatrix} Q_n^{(0)}(l_1'|l_1) & Q_n^{(0)}(l_1'|l_2) \\ Q_n^{(0)}(l_2'|l_1) & Q_n^{(0)}(l_2'|l_2) \end{bmatrix} \tag{2.6}$$

where $Q_n^{(0)}$ is given by (2.1) with

$$\phi(\theta) = \prod_{l=1}^{n} (w_{-1} \, \mathrm{e}^{-i\theta} + \mathrm{e}^{ix_l} + w_1 \, \mathrm{e}^{i\theta}). \tag{2.7}$$

Manipulation of (2.6) gives

$$Z_n^{(\mathrm{rt})}(l_1', l_2'|l_1, l_2) = \left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi} \mathrm{d}\theta_1 \int_0^{2\pi} \mathrm{d}\theta_2 \prod_{k=1}^{n} \int_0^1 \mathrm{d}x_k \, \mathrm{e}^{-ix_k}$$

$$\times \prod_{l=1}^{2} (w_{-1} \, e^{-i\theta_l} + \mathrm{e}^{ix_k} + w_1 \, \mathrm{e}^{i\theta_l}) \det[\mathrm{e}^{-i(l_j - l_k')\theta_l}]_{j,k=1,2}. \tag{2.8}$$

The integration over the x_k can now be done to give

$$Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N'|l_1, \ldots, l_N)$$

$$= \left(\frac{1}{2\pi}\right)^N \left(\prod_{k=1}^{N} \int_0^{2\pi} \mathrm{d}\theta_k \right) \left(\sum_{j=1}^{N} (w_{-1} \, \mathrm{e}^{-i\theta_j} + w_1 \, \mathrm{e}^{i\theta_j}) \right)^n$$

$$\times \det[\mathrm{e}^{-i(l_j - l_k')\theta_l}]_{j,k=1,\ldots,N} \tag{2.9}$$

with $N = 2$. By generalizing the above arguments for two walkers to N walkers, we arrive at the conjecture (2.9) for general N.

2.2. A multidimensional difference equation.

From the definition of the random turns model, it follows that the partition function is the unique solution of the multidimensional difference equation

$$\begin{aligned} Z_{n+1}^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) \\ &= w_1[Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1 - 1, l_2, \ldots, l_N) \\ &\quad + Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, l_2 - 1, l_3, \ldots, l_N) \\ &\quad + \ldots + Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_{N-1}, l_N - 1)] \\ &\quad + w_{-1}[Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1 + 1, l_2, \ldots, l_N) \\ &\quad + Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, l_2 + 1, l_3, \ldots, l_N) \\ &\quad + \ldots + Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_{N-1}, l_N + 1)] \end{aligned} \tag{2.10}$$

subject to the non-intersection condition

$$Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) = 0 \qquad \text{if } l_j = l_k \tag{2.11}$$

for any $1 \leq j, k \leq N$ $(j \neq k)$, and the initial condition

$$Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) = \prod_{k=1}^{N} \delta_{l_k', l_k}. \tag{2.12}$$

To verify that (2.9) satisfies (2.10) we note that (2.9) gives

$$\begin{aligned} Z_{n+1}^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) \\ &= \left(\frac{1}{2\pi}\right)^N \sum_{p=1}^{N} \prod_{k=1}^{N} \int_0^{2\pi} \mathrm{d}\theta_k (w_{-1} \mathrm{e}^{-i\theta_p} + w_1 \mathrm{e}^{i\theta_p}) \\ &\quad \times \left[\sum_{j=1}^{N} (w_{-1} \mathrm{e}^{-i\theta_j} + w_1 \mathrm{e}^{i\theta_j})\right]^n \det[\mathrm{e}^{-i(l_j - l_k')\theta_j}]_{j,k=1,\ldots,N} \end{aligned} \tag{2.13}$$

Since

$$\mathrm{e}^{\mp i\theta_p} \det[\mathrm{e}^{-i(l_j - l_k')\theta_j}]_{j,k=1,\ldots,N} = \det \begin{bmatrix} \mathrm{e}^{-i(l_{j_1} - l_k')\theta_{j_1}} \\ \mathrm{e}^{-i(l_p \pm 1 - l_k')\theta_p} \\ \mathrm{e}^{-i(l_{j_2} - l_k')\theta_{j_2}} \end{bmatrix}_{\substack{j_1 = 1, \ldots, p-1 \\ j_2 = p+1, \ldots, N \\ k = 1, \ldots, N}} \tag{2.14}$$

where j_1 and j_2 label the rows and k the columns, using (2.9) we can immediately identify the right-hand side of (2.13) with the right-hand side of (2.10) as required.

To verify (2.11) we simply note that if $l_j = l_k$ for any $j \neq k$ then two rows of the matrix in (2.9) are the same so the determinant vanishes.

Finally, to verify the initial condition (2.12) we first specify the initial ordering

$$l_1' < l_2' < \ldots < l_N'. \tag{2.15}$$

Since the walkers' paths cannot cross this means that the final positions also have the ordering

$$l_1 < l_2 < \ldots < l_N. \tag{2.16}$$

Now (2.9) gives

$$Z_0^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) = \left(\frac{1}{2\pi}\right)^N \left(\prod_{k=1}^{N} \int_0^{2\pi} \mathrm{d}\theta_k\right) \det[\mathrm{e}^{-i(l_j - l_k')\theta_j}]_{j,k=1,\ldots,N}. \tag{2.17}$$

Expanding the determinant as a sum over permutations allows the integrations in (2.17) to be performed to give

$$Z_0^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) = \sum_{P=1}^{N!} \varepsilon(P) \prod_{j=1}^{N} \delta_{l_j, l_{P(j)}'} \tag{2.18}$$

where the sum is over all the permutations of $1, \ldots, N$ and $\varepsilon(P)$ denotes their parity. Due to the orderings (2.15) and (2.16) all terms must vanish except for the identity permutation. Thus (2.12) is satisfied.

Since the difference equation, the initial condition and the boundary condition are satisfied by (2.9), we conclude that it is an exact formula for the partition function of the random turns model.

2.3. *The discrete heat equation in N dimensions*

An intrinsically interesting aspect of the difference equation (2.10) is that it can be transformed into a discrete version of the N-dimensional heat equation. To see this, simply subtract $Z_n(l_1', \ldots, l_N' | l_1, \ldots, l_N)$ from both sides of (2.10) and take

$$w_1 = w_{-1} = \frac{1}{2N}. \tag{2.19}$$

Then, as claimed, (2.10) becomes the discrete N-dimensional heat equation

$$\frac{1}{2} \sum_{j=1}^{N} D_{l_j}^2 Z_n(l_1', \ldots, l_N' | l_1, \ldots, l_N) = D_n Z_n(l_1', \ldots, l_N' | l_1, \ldots, l_N) \tag{2.20}$$

where $D_{l_j}^2$ denotes the difference operator

$$D_{l_j} f(l_j) = f(l_j + 1) + f(l_j - 1) - 2f(l_j) \tag{2.21}$$

and D_n denotes the difference operator

$$D_n g_n = g_{n+1} - g_n.$$

2.4. *Large n behaviour of the partition function*

Let us suppose $w_1 = w_{-1}$ and that the number of walkers N is fixed. For large-n and for any periodic function $f(\theta_1, \ldots, \theta_N)$ (period 2π in each θ_j),

$$\left(\prod_{k=1}^{N} \int_0^{2\pi} \mathrm{d}\theta_k \right) \left(\sum_{j=1}^{N} \cos \theta_j \right)^n f(\theta_1, \ldots, \theta_N)$$

$$\sim N^n \left(\prod_{k=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}\theta_k \, \mathrm{e}^{-n(\theta_k)^2/2N} \right) f(\theta_1, \ldots, \theta_N). \tag{2.22}$$

Using this asymptotic formula in (2.9) and explicitly performing the integration over the θ_j variables gives that for a large number n of steps,

$$Z_n^{(\mathrm{rt})}(l_1', \ldots, l_N' | l_1, \ldots, l_N) \sim \left(\frac{N}{2\pi n} \right)^{N/2} (2w_1 N)^n \det[\mathrm{e}^{-N(l_j - l_k')^2/2n}]_{j,k=1,\ldots,N}. \tag{2.23}$$

By writing

$$n/N = Dt \qquad 2w_1 N = \mathrm{e}^{-\sigma t} \qquad \text{and} \qquad l_j = x_j \tag{2.24}$$

we observe that the right-hand side of (2.23) is precisely the partition function for N impenetrable particles undergoing Brownian motion (see e.g. [5], equation (2.5)]). The large-n behaviour of probabilistic quantities such as the probability of survival and the probability of a reunion for walkers in the random turns model is thus identical to the large-t behaviour of the corresponding quantities in the Brownian motion model. The latter can be found in [2].

2.5. A counting problem

Suppose the number of walkers N is odd. Then analogous to the situation for the lock step model [7], it is easy to show that for a circle of M sites, the partition function is given by (2.9) with the replacements

$$\theta_j \mapsto 2\pi r_j/M \qquad \text{and} \qquad \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta_j \mapsto \frac{1}{M}\sum_{r_j=0}^{M-1}. \tag{2.25}$$

When $w_1 = 1$, $w_{-1} = 0$ and $M = n$, the partition function counts the number of walks, according to the rules of the random turn model with walkers moving to the right only, which start at $l'_1, \ldots, l'_N$ and after M steps on a circular lattice of M sites finish at $l'_2, l'_3, \ldots, l'_N, l'_1$ in that order. This counting problem occurs in a high temperature graphical expansion of the Hubbard model in the atomic limit [10].

Denoting the number of walks by $C_M(l'_1, \ldots, l'_N)$ we have from (2.9) and (2.25) that

$$C_M(l'_1, \ldots, l'_N) = \left(\prod_{j=1}^{N}\sum_{r_j=1}^{M}\right)\left(\sum_{p=1}^{N} \mathrm{e}^{2\pi i r_p/M}\right)^M \det[\mathrm{e}^{-2\pi i(l'_j - l'_{k+1})r_j/M}]_{j,k=1,\ldots,N} \tag{2.26}$$

where $l'_{N+1} := l'_1$. If we denote by S_M the sum of $C_M(l'_1, \ldots, l'_N)$ over all possible initial conditions

$$1 \leqslant l'_1 < l'_2 < \ldots < l'_N \leqslant M$$

we observe that S_M can be computed from (2.26) by dividing by $N!$ and summing each of the l'_j from 1 to M since the determinant is symmetric in each l'_j. Doing this and performing a simple manipulation of the determinant gives

$$S_M = \frac{1}{N!}\left(\prod_{j=1}^{N}\sum_{r_j=1}^{M}\right)\left(\sum_{p=1}^{N} \mathrm{e}^{2\pi i r_p/M}\right)^M \det[\delta_{r_p, r_{k-1}}]_{j,k=1,\ldots,N} \tag{2.27}$$

where $\delta_{a,b}$ denotes the Kronecker delta.

The matrix in (2.27) is symmetric in each r_j and vanishes if $r_j = r_{j'}$ for $j \neq j'$. Thus in the sum over the $r'_j s$ can be ordered so that

$$1 \leqslant r_1 < r_2 < \ldots < r_N \leqslant M \tag{2.28}$$

provided the expression (2.27) is multiplied by $N!$. The determinant is then easily seen to be equal to unity so we have

$$S_M = \sum_{1 \leqslant r_1 < \ldots < r_N \leqslant M} \left(\sum_{p=1}^{N} \mathrm{e}^{-2\pi i r_p/M}\right)^N \tag{2.29}$$

This formula agrees with that derived in [10] using Fermion methods.

3. Continuous domain walls in a strip

As we have previously remarked, if the lattice spacing tends to zero it is possible to adjust the other parameters so that the lock step and random turns model reduce to a model of impenetrable particles undergoing Brownian motion. This Brownian motion model can be interpreted as a model of continuous domain walls in a two-dimensional region, with time forming one of the space directions (see figure 3).

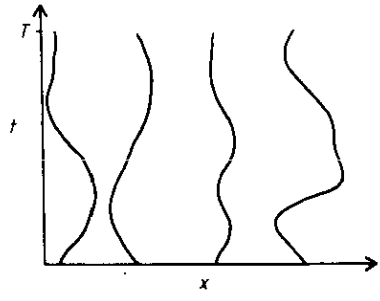

Figure 3. The correspondence between impenetrable Brownian motion particles and continuous domain walls. The time direction for the Brownian particles corresponds to a space direction for the domain walls.

Our objective in this section is to study the thermodynamics of continuous domain walls in a strip, and in particular to relate the mean spacing between walls ν to the (dimensionless) chemical potential σ. This will be done by first calculating the partition function for a particular general value of ν and then minimizing the corresponding free energy with respect to ν.

To calculate this partition function we will take the continuum limit of the partition function $Z_n^{(\mathrm{ls})}(N, M)$ calculated in [7, Eq. (2.28)] for the lock step model on a circular lattice of M sites with the initial and final configuration of the walkers equally spaced. We have

$$Z_n^{(\mathrm{ls})}(N, M) = \prod_{l=0}^{N-1} \frac{1}{\nu} \sum_{b=0}^{\nu-1} [\Phi(2\pi(l/M + b/\nu)]^n \tag{3.1}$$

where

$$\Phi(\theta) = \mathrm{e}^{-\sigma'} \cos\theta \tag{3.2}$$

and n denotes the number of steps (assumed even), N denotes the number of walkers (assumed odd) and we have chosen

$$w_1 = w_{-1} = \tfrac{1}{2}\mathrm{e}^{-\sigma'}. \tag{3.3}$$

To take the continuum limit a factor τ^{-N} (τ denotes the lattice spacing) must first be included and the limits

$$\begin{array}{llll} M \to \infty & l \to \infty & n \to \infty & \tau \to 0 \\ \tau M \to L & \tau^2 n \to Dt & \sigma' n \to -\sigma & \end{array} \tag{3.4}$$

taken, where L denotes the length of the system, t can be interpreted as the width and σ is the chemical potential. (Note: we have replaced σ as used in [7] by $-\sigma$ to allow identification with the chemical potential.)

Since in the limit (3.4)

$$\cos^n(2\pi(l/M + b/\nu)) \to \mathrm{e}^{-2\pi^2(l+bN)^2 Dt/L^2} \tag{3.5}$$

we obtain

$$Z_n^{(\mathrm{ls})}(N, M) \to Z_t^{(\mathrm{Bm})}(N, L) = \nu^{-N} \mathrm{e}^{\sigma t N} \prod_{l=0}^{N-1} \sum_{b=-\infty}^{\infty} \mathrm{e}^{-2\pi^2(l+bN)^2 Dt/L^2} \tag{3.6}$$

where the superscript Bm refers to Brownian motion. Applying the Poisson summation formula gives

$$\sum_{b=-\infty}^{\infty} \mathrm{e}^{-2\pi^2(l+bN)^2 Dt/L^2} = \left(\frac{\nu^2}{2\pi Dt}\right)^{1/2} \theta_3(\pi l/N; \mathrm{e}^{-\nu^2/2Dt}) \tag{3.7}$$

where

$$\theta_3(z; q) = \sum_{n=-\infty}^{\infty} q^{n^2} \mathrm{e}^{2inz} = \prod_{n=1}^{\infty} (1+q^{2n-1}\mathrm{e}^{2iz})(1+q^{2n-1}\mathrm{e}^{-2iz})(1-q^{2n}). \tag{3.8}$$

Using the product expansion (3.8) in (3.7) and substituting this result in (3.6) allows the product in (3.6) to be performed via the identity

$$\prod_{l=0}^{N-1} (1 - a\,\mathrm{e}^{2\pi i l/N}) = (1 - a^N). \tag{3.9}$$

This procedure gives the evaluation

$$Z_t^{(\mathrm{Bm})}(N, L) = (2\pi Dt)^{-N/2} \mathrm{e}^{\sigma t N} \theta_3(0, \mathrm{e}^{-\nu^2 N/2Dt}) \prod_{n=1}^{\infty} \frac{(1-\mathrm{e}^{-\nu^2 n/Dt})^N}{(1-\mathrm{e}^{-\nu^2 nN/Dt})}. \tag{3.10}$$

The strip free energy $f_t(\nu)$ per unit length is defined as

$$\beta f_t(\nu) := -\lim_{L\to\infty} \frac{1}{L} \log Z_t^{(\mathrm{Bm})}(N, L) \tag{3.11}$$

where $L/N := \nu$ and t are held fixed. From (3.10) we have

$$\beta f_t(\nu) = \frac{1}{\nu}\left(\tfrac{1}{2}\log(2\pi Dt) - \sigma t - \log \prod_{n=1}^{\infty} (1-\mathrm{e}^{-\nu^2 n/Dt})\right). \tag{3.12}$$

Now the final term in (3.12) is always positive and tends to zero as $\nu \to \infty$. Thus we see that if

$$\sigma < \frac{1}{2t}\log(2\pi Dt) \tag{3.13}$$

then $f_t(\nu)$ is minimized by $\nu \to \infty$. Let us denote the right hand side of (3.13) by σ_c and suppose $\sigma = \sigma_c + \varepsilon$ where ε is positive and small. Expanding the logarithm to first order for large ν then gives

$$\beta f_t(\nu) \sim \frac{1}{\nu}(-\varepsilon t + \mathrm{e}^{-\nu^2/Dt}) \tag{3.14}$$

which is minimized when

$$\varepsilon t \sim (2\nu^2/Dt)\,\mathrm{e}^{-\nu^2/Dt}. \tag{3.15}$$

The equation (3.15) specifies the divergence of the average spacing between walls ν as the critical chemical potential is approached.

4. Dislocations

Dislocations occur when two domain walls meet (or come into close proximity) and end without crossing the whole system (see figure 4). The logarithm of the correlation between dislocations in the bulk gives their effective potential (see e.g. [11]). In this section we seek to calculate these correlations by using the Brownian motion model of continuous domain walls.

4.1. Notation and definitions

Let us denote by

$$F_1(x'_1, \ldots, x'_N; x_a, t_a; x''_3, \ldots, x''_N; t_b) \tag{4.1}$$

the partition function for N continuous domain walls with a defect originating from the bottom of the system. The N walls start at $x'_1, \ldots, x'_N$ along the bottom of the system ($t = 0$) with the defect occurring at $x = x_a$ when $t = t_a$. The remaining $N-2$ walls end at $x''_3, \ldots, x''_N$ when $t = t_b$ (see figure 4(a)).

Let

$$H(x_b, t_b; x''_3, \ldots, x''_N; x'_1, \ldots, x'_N; T) \tag{4.2}$$

denote the partition function for N continuous domain walls with a defect orginating from the top of the system. The N walls are attached at $x'_1, \ldots, x'_N$ along the top of the system ($t = T$) with the defect occurring at $x = x_b$ when $t = t_b$. The remaining $N-2$ walls also end when $t = t_b$ at the positions $x''_3, \ldots, x''_N$ (see figure 4(b)).

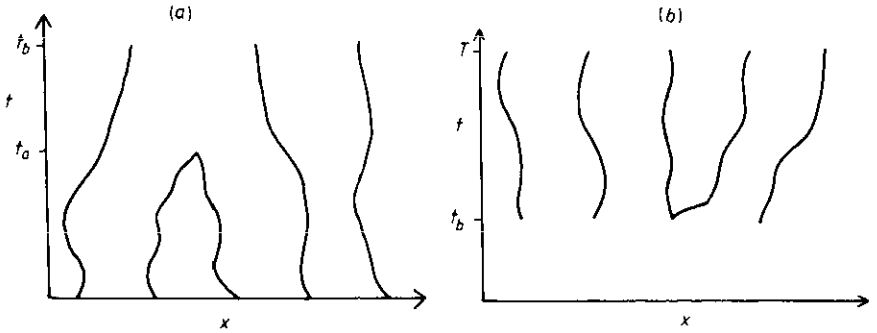

Figure 4. The two different types of defects: (a) originating from the bottom of the system and (b) originating from the top.

Analogous to (4.1), let

$$F_2(x'_1, \ldots, x'_N; x_a, t_a; x_b, t_b; x''_5, \ldots, x''_N; t_b) \tag{4.3}$$

denote the partition function for N continuous domain walls with two defects originating from the bottom of the system. The defects occur at $x = x_a$ when $t = t_a$ and at $x = x_b$ when $t = t_b$.

Finally, as we have done previously [7], let

$$G(x'_1, \ldots, x'_N | x_1, \ldots, x_N; t') \tag{4.4}$$

denote the partition function for N continuous domain walls which start at $x'_1, \ldots, x'_N$ when $t = 0$ and finish at $x_1, \ldots, x_N$ when $t = t'$.

With this notation, the correlation $\rho^{(u)}(x^*, t^*)$ between two unlike dislocations in the bulk is defined as

$$\rho^{(u)}(x^*, t^*) = \lim G^{-1}(x'_1, \ldots, x'_N | x'_1, \ldots, x'_N; T) \times \prod_{l=3}^{N} \left(\int_0^L \mathrm{d}x_l \right) F_1(x'_1, \ldots, x'_N; x_a, t_a; x_3, \ldots, x_N; t_b) \times H(x_b, t_b; x_3, \ldots, x_N; x'_1, \ldots, x'_N; T) \tag{4.5}$$

where the limit is

$$L, N, T, t_a, t_b \to \infty \qquad t_b - t_a := t^* \qquad x_b - x_a := x^* \qquad \text{and} \qquad L/N = \nu \text{ all fixed.} \tag{4.6}$$

We take

$$x'_j = jL/N \qquad j = 1, 2, \ldots, N \tag{4.7}$$

so that along the top and bottom of the system the walls are equally spaced.

Similarly, the correlation $\rho^{(l)}(x^*, t^*)$ between two like dislocations in the bulk is defined as

$$\rho^{(l)}(x^*, t^*) = \lim G^{-1}(x'_1, \ldots, x'_N | x'_1, \ldots, x'_N; T) \times \prod_{l=5}^{N+2} \left(\int_0^L \mathrm{d}x_l \right) F_2(x'_1, \ldots, x'_{N+2}; x_a, t_a; x_b, t_b; x_5, \ldots, x_{N+2}; t_b) \times G(x_5, \ldots, x_{N+2} | x'_5, \ldots, x'_{N+2}; T - t_b) \tag{4.8}$$

where the limit is specified by (4.6) and the $N+2$ walls along the bottom are taken to be equally spaced between 0 and L, and similarly the $N-2$ walls along the top.

The choice of normalization in (4.5) and (4.8) is based on matching the number of walls in the reference (normalizing) system with the number of walls in the test (defect) system, where in the latter each defect counts for a wall. Also, in the definition (4.8) of $\rho^{(l)}$, to obtain a finite non-zero correlation for large-t, it is necessary that the chemical potential of the test system differ by $O(1/N^2)$ from that of the reference system.

4.2. Determinant formulas for G, F_1, G_2 and H

In [7, equation (4.15)] we have shown that provided N is odd and assuming periodic boundary conditions

$$G(x'_1, \ldots, x'_N | x_1, \ldots, x_N; t) = \det[Q_t(x_j | x'_k)]_{j,k=1,\ldots,N} \tag{4.9}$$

where

$$Q_t(x|x') = \frac{\mathrm{e}^{\sigma t}}{L} \theta_3(\pi(x - x')/L; \mathrm{e}^{-2\pi^2 Dt/L^2}). \tag{4.10}$$

(Again we have replaced σ used in [7] by $-\sigma$ to allow identification with the chemical potential.)

From the definitions of G and F_1 we see that F_1 can be expressed in terms of G according to the formula

$$F_1(x'_1, \dots, x'_N; x_a, x_{a'}, t_a; x''_3, \dots, x''_N; t_b) = \prod_{l=3}^{N} \int_0^L dx_l \, G(x'_1, \dots, x'_N | x_a, x_{a'}, x_3, \dots, x_N; t_a) \times G(x_3, \dots, x_N | x''_3, \dots, x''_N; t_b - t_a) \tag{4.11}$$

where F_1 as defined here has the ends of the defect a distance $|x_a - x_{a'}|$ apart in the X-direction. Using (4.9) and the series form (3.8) of $\theta_3(z; q)$ and by expanding out the determinant as a sum over permutations, the integrations in (4.11) can readily be performed to give

$$F_1(x'_1, \dots, x'_N; x_a, x_{a'}, t_a; x''_3, \dots, x''_N; t_b) = \det \begin{bmatrix} Q_{t_a}(x_a | x'_k) \\ Q_{t_a}(x_{a'} | x'_k) \\ Q_{t_b}(x''_j | x'_x) \end{bmatrix}_{\substack{j=3,\dots,N \\ k=1,\dots,N}} \tag{4.12}$$

where the entries in the first two rows have a distinct form from the remaining rows and thus have been written separately.

To calculate the correlation between defects it is first desirable to eliminate the finite separation in (4.11) between the ends of a single defect. However, this requires an infinite amount of energy and can only be done if (4.11) is renormalized by dividing through by $x_{a'} - x_a$. Doing this in (4.12) allows the limit $x_{a'} \to x_a$ to be taken to give

$$F_1(x'_1, \dots, x'_N; x_a, t_a; x''_3, \dots, x''_N; t_b) = \det \begin{bmatrix} Q_{t_a}(x_a | x'_k) \\ (\partial/\partial x_a) Q_{t_a}(x_a | x'_k) \\ Q_{t_b}(x''_j | x'_k) \end{bmatrix}_{\substack{j=3,\dots,N \\ k=1,\dots,N}} \tag{4.13}$$

Similar arguments give

$$F_2(x'_1, \dots, x'_N; x_a, t_a; x''_5, \dots, x''_N; t_b) = \det \begin{bmatrix} Q_{t_a}(x_a | x'_k) \\ (\partial/\partial x_a) Q_{t_a}(x_a | x'_k) \\ Q_{t_b}(x_b | x'_k) \\ (\partial/\partial x_b) Q_{t_b}(x_b | x'_k) \\ Q_{t_b}(x''_j | x'_k) \end{bmatrix}_{\substack{j=5,\dots,N \\ k=1,\dots,N}} \tag{4.14}$$

and

$$H(x_b, t_b; x''_3, \dots, x''_N; x'_1, \dots, x'_N; T) = \det \begin{bmatrix} Q_{T-t_b}(x_b | x'_k) \\ (\partial/\partial x_b) Q_{T-t_b}(x_b | x'_k) \\ Q_{T-t_b}(x''_j | x'_k) \end{bmatrix}_{\substack{j=3,\dots,N \\ k=1,\dots,N}} \tag{4.15}$$

4.3. Large time behaviour of G, F_1, F_2 and H

To calculate the correlations (4.5) and (4.8) we only require the large time behaviours of G, F_1, F_2 and H. Providing we take x'_j ($j = 1, \dots, N$) to be given by (4.7), these

asymptotic forms can easily be deduced from the above determinant formulas by first multiplying by unity in the form of

$$i^{-N(N-1)/2}N^{-N/2}\det[\mathrm{e}^{2\pi ilk/N}]_{\substack{l=1,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.16}$$

(see e.g. [12] for a derivation of this result). For example, multiplying (4.9) by (4.16) gives

$$G(x'_1,\ldots,x'_N|x''_1,\ldots,x''_N;\,t)$$
$$= A_N(t)\det[\mathrm{e}^{2\pi ix_l k/L}\theta_3(\pi Nx_j/L+i\pi^2 DtNk/L^2;\,\mathrm{e}^{-2\pi^2 DtN^2/L^2}]_{\substack{j=1,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.17}$$

where

$$A_N(t) = i^{-N(N-1)/2}L^{-N}N^{N/2}\,\mathrm{e}^{\sigma Nt}\,\mathrm{e}^{-\pi^2 DtN(N^2-1)/6} \tag{4.18}$$

(we have used the result $\Sigma_{k=-(N-1)/2}^{(N-1)/2}k^2 = N^2(N-1)/12$). For large-$t$ the theta function in (4.17) tends to unity and thus

$$G(x'_1,\ldots,x'_N|x''_1,\ldots,x''_N;\,t)\sim A_N(t)\det[\mathrm{e}^{2\pi ix_j k/L}]_{\substack{j=1,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.19}$$

A similar procedure applied to (4.13)–(4.15) shows

$$F_1(x'_1,\ldots,x'_N;\,x_a,\,t_a\,;\,x''_3,\ldots,x''_N;\,t_b)$$
$$\sim A_N(t_b)\det\begin{bmatrix} \mathrm{e}^{-\sigma t^*}\,\mathrm{e}^{2\pi^2 Dt^*k^2/L^2}\,\mathrm{e}^{2\pi ix_a k/L} \\ (2\pi ik/L)\,\mathrm{e}^{-\sigma t^*}\,\mathrm{e}^{2\pi^2 Dt^*k^2/L^2}\,\mathrm{e}^{2\pi ix_a k/L} \\ \mathrm{e}^{2\pi ix_j k/L}\end{bmatrix}_{\substack{j=3,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.20}$$

$$F_2(x'_1,\ldots,x'_N;\,x_a,\,t_a\,;\,x''_5,\ldots,x''_N;\,t_b)$$
$$\sim A_N(t_b)\det\begin{bmatrix} \mathrm{e}^{-\sigma t^*}\,\mathrm{e}^{2\pi^2 Dt^*k^2/L^2}\,\mathrm{e}^{2\pi ix_a k/L} \\ (2\pi ik/L)\,\mathrm{e}^{-\sigma t^*}\,\mathrm{e}^{2\pi^2 Dt^*k^2/L^2}\,\mathrm{e}^{2\pi ix_a k/L} \\ \mathrm{e}^{2\pi ix_b k/L} \\ (2\pi ik/L)\,\mathrm{e}^{2\pi ix_b k/L} \\ \mathrm{e}^{2\pi ix_j k/L}\end{bmatrix}_{\substack{j=5,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.21}$$

and

$$H(x_b,\,t_b\,;\,x''_3,\ldots,x''_N;\,x'_1,\ldots,x'_N;\,T)$$
$$\sim A_N(T-t_b)\det\begin{bmatrix} \mathrm{e}^{2\pi ix_b k/L} \\ (2\pi ik/L)\,\mathrm{e}^{2\pi ix_b k/L} \\ \mathrm{e}^{2\pi ix_j k/L}\end{bmatrix}_{\substack{j=3,\ldots,N\\ k=-(N-1)/2,\ldots,(N-1)/2}} \tag{4.22}$$

where $A_N(t)$ is given by (4.18) and t^* by (4.6).

4.4. The equilibrium value of ν in terms of σ.

The partition function $Z_t^{(\mathrm{Bm})}(N, L)$ for the Brownian motion model of continuous domain walls is given by (3.6). The form (3.6) allows the large-t behaviour to be easily deduced as

$$Z_t^{(\mathrm{Bm})}(N, L)\sim \nu^{-N}\,\mathrm{e}^{\sigma tN-\pi^2 Dt(N-1/N)/6\nu^2} \tag{4.23}$$

where the summation formula noted after (4.18) has been used. The bulk free per unit area energy f is thus

$$\beta f := -\lim_{L,t\to\infty} \frac{1}{Lt} \log Z_t^{(\mathrm{Bm})}(N, L) = \left(-\frac{\sigma}{\nu} + \frac{\pi^2 D}{6\nu^3} \right) \tag{4.24}$$

where $\nu := L/N$, which is at a minimum when

$$\sigma = \frac{\pi^2 D}{2\nu^2}. \tag{4.25}$$

Thus a striped phase requires the chemical potential σ to be positive and (4.25) specifies the mean spacing between walls in terms of σ.

4.5. *Coloumb gas analogue for $t^* = 0$*

The correlations between defects have a Coulomb gas analogue if the defects are on the same level in the t direction so that $t^* = 0$. To see this, we note that

$$i^{-N(N-1)/2} \det[e^{2\pi i x_j k/L}]_{\substack{j=1,\dots,N \\ k=-(N-1)/2,\dots,(N-1)/2}}$$

$$= 2^{N(N-1)/2} \prod_{1\le j<k\le N} \sin \pi(x_k - x_j)/L \tag{4.26}$$

which follows from the van der Monde determinant formula

$$\det[x_j^{k-1}]_{j,k=1,\dots,N} = \prod_{1\le j<k\le N} (x_k - x_j). \tag{4.27}$$

Using (4.26) and where necessary confluent variations we find from (4.19)–(4.22) that when $t^* = 0$

$$F_1 H = |A_N(t_b) A_N(T - t_b)| 2^{N(N-1)} \left(\frac{\pi}{L}\right)^2$$

$$\times \prod_{k=3}^{N} [\sin^2 \pi(x_k - x_a)/L][\sin^2 \pi(x_k - x_b)/L]$$

$$\times \prod_{3\le j<k\le N} \sin^2 \pi(x_k - x_j)/L \tag{4.28}$$

$$F_2 G = |A_{N+2}(t_b) A_{N-2}(T - t_b)| 2^{N^2-N+4} \left(\frac{\pi}{L}\right)^2 \sin^4 \pi(x_k - x_j)/L$$

$$\times \prod_{k=5}^{N+2} [\sin^2 \pi(x_k - x_a)/L][\sin^2 \pi(x_k - x_b)/L]$$

$$\times \prod_{5\le j<k\le N+2} \sin^2 \pi(x_k - x_j)/L \tag{4.29}$$

and

$$G^2 = |A_N(t_b) A_N(T - t_b)| 2^{N(N-1)} \prod_{1\le j<k\le N} \sin^2 \pi(x_k - x_j)/L. \tag{4.30}$$

In (4.28) and (4.29) the arguments of $F_1 H$ and $F_2 G$ are the same as in (4.5) and (4.8) respectively. We have included a formula for G^2 since

$$G(x_1', \dots, x_N' | x_1', \dots, x_N') = \prod_{l=1}^{N} \int_0^L \mathrm{d}x_l (G(x_1', \dots, x_N' | x_1, \dots, x_N))^2 \tag{4.31}$$

and thus (4.30) and (4.31) can be used in the definitions of the correlation (4.5) and (4.8).

Using (4.28) in the numerator of (4.5) and (4.31) and (4.30) in the denominator shows that

$$(L/\pi)^2 [\sin^2 \pi(x_b - x_a)/L] \rho^{(u)}(x^*, 0) \tag{4.32}$$

is precisely the definition of the correlation between two particles of charge q at the points x_a and x_b in a one-component log-potential Coulomb gas on a line at the coupling $\Gamma := q^2/k_B T = 2$. As such, for large separations between the test charges, to leading order it must equal $1/\nu^2$. Hence we have that in the limit $L \to \infty$ and for $x_b - x_a := x^*$ large,

$$\rho^{(u)}(x^*, 0) \sim \frac{1}{(\nu x^*)^2}. \tag{4.33}$$

To obtain a similar result for the correlation between like defects we must first choose the mean wall spacing ν in the test system to be specified by

$$\sigma = \frac{\pi^2 D}{2\nu^2}\left(1 + \frac{2}{N^2}\right) \tag{4.34}$$

(in the reference system it is specified by (4.25)). Then, provided $T - 2t_b$ is finite, in the large time limit the ratio

$$\left| \frac{A_{N+2}(t_b) A_{N-2}(T - t_b)}{A_N(t_b) A_N(T - t_b)} \right| \tag{4.35}$$

is non-zero and finite. If L is also large we readily deduce from (4.18) that (4.35) tends to unity. Under these circumstances we see by substituting (4.29) in the numerator of (4.8) and (4.31) and (4.30) in the denominator that

$$\left(\frac{L}{4\pi}\right)^2 \frac{1}{\sin^2 \pi(x_b - x_a)/L} \rho^{(l)}(x^*, 0) \tag{4.36}$$

is precisely the same correlations as (4.32). Thus, for large-L and large $|x_b - x_a|$ (but $|x_b - x_a|/L$ still small) we have

$$\rho^{(l)}(x^*, 0) \sim \frac{1}{\nu^2}\left(\frac{2\pi}{L}\right)^4 (x^*)^2. \tag{4.37}$$

4.6. The correlations for general t^*

To evaluate $\rho^{(u)}(x^*, t^*)$ we substitute the asymptotic formulas (4.20) and (4.22) for the integrand in (4.5) (it is convenient to first take the complex conjugate of (4.22), which is valid since H is real). By expanding each determinant as a sum over permutations the multidimensional integral in (4.5) can easily be performed due to the orthogonality on $[0, L]$ of $\{e^{2\pi i x k/L}\} k \in Z$. The resulting expression is a Riemann sum approximation to a double integral.

The denominator in (4.5) is precisely $Z_{\beta}^{(\mathrm{Bm})}(N, L)$ and is thus given asymptotically by (4.23). Substituting (4.25) for σ in the numerator allows the limit (4.6) to be taken to obtain

$$\rho^{(u)}(x^*, t) = \frac{2\pi^2}{\nu^4} \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \mathrm{d}r\, \mathrm{d}s (r-s)^2 \, \mathrm{e}^{2\pi^2 Dt^*(r^2+s^2-1/2)/\nu^2} \, \mathrm{e}^{2\pi i x^*(r+s)/\nu}. \tag{4.38}$$

For t^* and x^* large the integrand takes its maximum value for $r = -s = \frac{1}{2}, -\frac{1}{2}$. Using Laplace's method we thus find from (4.38) that

$$\rho^{(u)}(x^*, t^*) \sim \frac{1}{\nu^2} \frac{1}{(x^*)^2 + (\pi D t^*/\nu)^2} \tag{4.39}$$

which agrees with (4.33) when $t^* = 0$.

To evaluate $\rho^{(l)}(x^*, t^*)$ we substitute the asymptotic formulas (4.21) and (4.19) for the integrand in (4.8), where for convenience the complex conjugate of (4.19) is used. As in the evaluation of $\rho^{(u)}(x^*, t^*)$, expanding each determinant as a sum over permutations allows the integrations in (4.8) to be performed. For the chemical potential of the test system we use (4.34) and for the denominator of (4.8) we substitute (4.23). If we take the ratio (4.35) as unity (which is valid for large t and L) we thus find

$$\rho^{(l)}(x^*, t^*) = \left(\frac{4\pi}{L}\right)^2 (N^2 - 1) \, \mathrm{e}^{-\pi^2 Dt^*/L^2} \sin^2(\pi x^*/L + i\pi^2 Dt^* N/L^2). \tag{4.40}$$

For large L, x^*, t^* (but x^*/L, $t^*/L \ll 1$) this behaves as

$$\rho^{(l)}(x^*, t^*) \sim \left(\frac{2\pi}{L}\right)^4 \frac{1}{\nu^2} [(x^*)^2 + (\pi D t^*/\nu)^2] \tag{4.41}$$

which, for $t^* = 0$, agrees with (4.37).

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